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TOWARDS A UNIVERSAL LAW OF TREE MORPHOMETRY BY COMBINING FRACTAL GEOMETRY AND STATISTICAL PHYSICS

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This article aims at establishing a very general law of plant organization. By introducing the notion of hydraulic lengths which are considered as the coordinates of a symbolic space with n -dimensions, a reasoning of statistical physics, derived from Maxwell's method, and combining with the fractal geometry leads to a law of hydraulics lengths distribution which could appear very general because it is the remarkable gamma law form

1 Introduction

1.1 The applications of morphometry in geomorphology

Before the conception of the fractal geometry by Mandelbrot (1975)¹, morphometric analysis was at first used by geologists to understand the river systems organization. Horton (1945)¹² links talweg sections by their source point and by their confluence point with an other talweg of similar importance. Horton defines two empirical laws expressed by two ratios:

- the bifurcation ratio, $R_C = \frac{N_{i-1}}{N_i}$, which has a constant value between 3 and 5 for river systems. N_i is the number of i order sections,
- the length ratio, $R_L = \frac{L_i}{L_{i-1}}$, which has a constant value between 1.5 and 3.5 for the rivers. L_i is the average length of i order sections.

Finally La Barbera and Rosso (1982)¹⁴ define fractal dimension for a drainage basin, $D = \frac{\ln R_C}{\ln R_L}$. Weibel and Gomez (1962)¹⁵ used morphometry to model lungs, then numerous studies were carried out on trees.

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1.2 Applications of the morphometry and fractal geometry on plants

Fitter (1982)¹⁶ presents a morphometric tree classification inspired by river networks. In his methodological study on root systems of herbaceous species, he shows that one can use Horton's laws by ordering ramifications according to the morphometric order to quantify the root ramification.

Holland (1969)¹⁷ shows that the ramification of several species of Eucalyptus can be described and be explained by Horton's laws and by the effect of the apical control in the young twigs' growth. Leopold (1971)¹⁸ working on different architecture plant (*Abies concolor*, *Pinus taeda*) comes to the same conclusion. He adds that the most likely classification seems to minimize the total length of the branches in the ramification system. Oohata and Shidei (1971)¹⁹ study with the aid of Horton's method the ramification of four types of ligneous plants among which shrubs with big evergreen leaves (*Cinnamomum camphora*) and conifers with evergreen leaves. He shows that the ramification ratio varies in a range much wider than river systems : from 3.0 to 8.0. This ratio varies according to the plant biologic type. Whitney (1976)²⁰ shows on 16 ligneous species that the ramification ratio depends mainly on the leaves disposal, on the deciduousity of the leaf and branches and on the needles size, and that it is more characteristic of species and relatively independent of external conditions.

Using the morphometric tree of Strahler (1958)¹³ shows that for the birch and the apple tree, the logarithms of the average numbers of terminal branches of every order of ramification, of the average diameter and of the number of buds carried by these branches are aligned compared to the ramification order. The logarithms of the twigs' average length are much more scattered. They deduct that these two species have a fractal ramification and that lengths are more significant of the specific shape of trees. Crawford and Young (1990)²¹ show, for oaks (*Quercus robur*) that the branches' distribution lengths follow a simple fractal algorithm. Berger (1991)²² uses fractals to model the growth of trees (*ficus elastica*), Chen *et al.* (1993)²³ to model the canopy of a poplar population (*Populus sp.*), Macmahon and Kronauer (1976)²⁴ to model the mechanics of the tree (*Quercus rubra*).

1.3 The invariant structure of plants

Generally, a branching system is constituted by the subset of branching systems. A branch is the part of a tree included between two successive ramifications. To study the branching organization, we shall use the typology of Strahler (1952)⁴ (see Figure 1):

- a bud or a growing shoot is called the first-order branch
- when two branches of order i join, a branch of order $i + 1$ is created,
- when two branches of different orders join, the branch immediately at the junction retains the higher of the two joining branches.

The branching system order is thus the main order found in the plant.

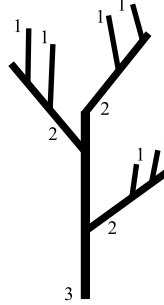


Figure 1. Tree typology and principle of the orders numerotation.

1.4 A universal law of morphology of landscapes

Two attempts have been made to apply a reasoning of statistical physics to hydrography. Lienhardt (1964)⁶ is the first to have perceived the interest of the statistical physics and Shreve (1966)⁷ has opened an innovative way by making the hypothesis that the law of the stream numbers as a function of the order results from a statistic of a large number of channels branching out at random, as the ideal gas law results from a statistics of a huge amount of molecules colliding at random. Like Mandelbrot (1975)¹ we are convinced that in both geomorphology and biomorphology, a statistical approach can be fruitful. However, one must be sure to respect two conditions that are basic ideas of statistical physics: *i*) the system size must be very large compared to the one of the constituent element that will be taken into account, *ii*) the local properties of the system must be homogeneous enough. The validity limits of the law that we are going to establish now is probably very closely linked to the respecting of these two conditions.

2 Demonstration of the law

2.1 Choice of the symbolic space

The difficulty of the extension of such a reasoning in the morphology of trees lies in the choice of the symbolic space as defined by Maxwell (Sears, 1971¹¹). The idea of our approach is based on the use of the symbolic space where the velocity components v_x , v_y , v_z , are replaced by *ad hoc* components. Maxwell uses a symbolic space, called velocities space, where each velocity vector ends in a point which is characterized by its coordinates v_x , v_y , v_z . He defines a function of these three coordinates:

$$F(v_x, v_y, v_z) = \frac{d^3N}{N dv_x dv_y dv_z} \quad (1)$$

Where d^3N is the number of molecules whose velocity vector leads to the elementary volume $dv_x dv_y dv_z$, among a total number of N molecules. From the very beginning of the reasoning we decided to take into account the fractal property

of our particular system, the branching system, by introducing two differences in comparison with Maxwell's symbolic space:

- we do not use a velocity space which would have no meaning here but a symbolic space, hydraulic length of the plant. We decide to call component of order i , the length of the set of links or part of links having the same order i (we shall note it l_i). So, for any point of the branching system, the hydraulic length is the sum $L = \sum l_i$ of its n -constituents. Where n is the order of the branching system. One can introduce the ratio:

$$r_l = \frac{\overline{l_i}}{\overline{l_{i-1}}} \quad (2)$$

Where numerator and denominator represent respectively the average of all the constituents with order i and the constituents with order $i + 1$.

- since we consider that each possible hydraulic length has n -constituents l_i , our symbolic space will no longer have three dimensions, as those of Maxwell, but n -dimensions, n being the order of the branching system. However, to use without any trouble the properties of n -dimensions vectorial space, instead of considering the components l_i , we will consider using their square roots $x_i = \sqrt{l_i}$, $L = X^2 = \sum_{i=1}^n x_i^2$. Thus, if we denote by N the total number of hydraulic length, we can define a function $F(l_1, l_2, \dots, l_n)$ with n -variables l_1, l_2, \dots, l_n :

$$F(l_1, l_2, \dots, l_n) = \frac{d^n N}{N dl_1 dl_2 \dots dl_n} \quad (3)$$

2.2 Choice of fundamental hypotheses

We adopt the same hypotheses as Maxwell, but by adapting them to our symbolic space and by taking into account consequences of the scalling invariance:

- according to the hypothesis of the independence of the hydraulic length distribution law, the components l_i are independent. One is so led to express function F as a product of n one-variable functions:

$$F(l_1, l_2, \dots, l_n) = f_1(l_1) f_2(l_2) \dots f_n(l_n) \quad (4)$$

- as Maxwell did for the velocities distribution we should admit that the distribution law of x_i is isotropic. According to scalling invariance and the relation 2, the i^{th} order component is on average r_l times larger than the $(i - 1)^{th}$ component. The hypothesis of isotropy must therefore not be applied to the symbolic space of coordinates $x_i = \sqrt{l_i}$, but z_i defined as reduced hydraulic lengths components $z_i = \frac{l_i}{r_l^{i-1}}$:

$$z_i = \sqrt{\frac{l_i}{r_l^{i-1}}} = \frac{x_i}{r_l^{\frac{i-1}{2}}} \quad (5)$$

So, the corresponding vector magnitude is Z such as:

$$Z^2 = \sum_{i=1}^n z_i^2 \quad (6)$$

The isotropy hypothesis entails that the density of points representing the vectors ends in the symbolic space has a spherical symmetry. Thus, we can consider that all the functions $f_i(l_i)$ in the relation 4 are identical provided that F is written as following:

$$F(l_1, l_2, \dots, l_n) = f(l_1)f\left(\frac{l_2}{r_l}\right) \dots f\left(\frac{l_n}{r_l^{n-1}}\right) \quad (7)$$

That is:

$$\phi(z_1, z_2, \dots, z_n) = \varphi(z_1)\varphi(z_2) \dots \varphi(z_n) \quad (8)$$

The function ϕ can also be written:

$$\phi(z_1, z_2, \dots, z_n) = \frac{d^n N}{N dz_1 dz_2 \dots dz_n} \quad (9)$$

If one moves in the surface of the hypersphere whose equation is:

$$z_1^2 + z_2^2 + \dots + z_n^2 = C \quad (10)$$

One has:

$$\phi(z_1, z_2, \dots, z_n) = C \quad (11)$$

C being a constant. These two above hypotheses are sufficient to determine the probability density function (pdf) of the hydraulic lengths.

2.3 Determination of the hydraulic lengths pdf

By taking the derivative of relations 10 and 11 gives:

$$2z_1 dz_1 + 2z_2 dz_2 + \dots + 2z_n dz_n = 0 \quad (12)$$

and

$$\frac{\partial \phi}{\partial z_1} dz_1 + \frac{\partial \phi}{\partial z_2} dz_2 + \dots + \frac{\partial \phi}{\partial z_n} dz_n = 0 \quad (13)$$

According to the relation 8:

$$\frac{1}{\phi} \frac{\partial \phi(z_i)}{\partial z_i} = \frac{1}{\varphi(z_i)} \frac{d\varphi(z_i)}{dz_i} \quad \forall i, 1 \leq i \leq n \quad (14)$$

Which allows us to replace 13 by:

$$\frac{1}{\varphi(z_1)} \frac{d\varphi(z_1)}{dz_1} + \frac{1}{\varphi(z_2)} \frac{d\varphi(z_2)}{dz_2} + \dots + \frac{1}{\varphi(z_n)} \frac{d\varphi(z_n)}{dz_n} = 0 \quad (15)$$

In this stage of the reasoning, we can use the Lagrange's optimization method (Bruhat, 1968¹⁰, Sears, 1971¹¹). It allows us to combine relations 12 and 15 by

multiplying 12 by a constant λ and by adding it in the relation 15. This leads to the following equation in which n differentials can be considered as independent:

$$\left[2\lambda z_1 \frac{1}{\varphi(z_1)} \frac{d\varphi(z_1)}{dz_1} \right] dz_1 + \dots + \left[2\lambda z_n \frac{1}{\varphi(z_n)} \frac{d\varphi(z_n)}{dz_n} \right] dz_n = 0 \quad (16)$$

All the expressions between brackets are simultaneously equal to zero, so that each one can be integrated:

$$\frac{1}{\varphi(z_i)} \frac{d\varphi(z_i)}{dz_i} = -2\lambda z_i \quad \forall i, 1 \leq i \leq n \quad (17)$$

Whose integral is:

$$\varphi(z_i) = A e^{-\lambda z_i^2} \quad (18)$$

Where z_i is distributed according to a normal law. The constant A can be calculated because the complete integral of φ between zero and the infinity must be equal to unity:

$$\int_0^\infty \varphi(z_i) dz_i = \int_0^\infty A e^{-\lambda z_i^2} dz_i = 1 \quad (19)$$

The variables change, by taking into account the relation 5, one sees then appearing the pdf of l_i :

$$f(l_i) = A e^{-\lambda \frac{l_i}{r_i^{i-1}}} \frac{l_i^{-\frac{1}{2}}}{2\sqrt{r_i^{i-1}}} \quad (20)$$

The graph of this law is a convex shape and quickly decreasing, more exactly, it is about a gamma law of general equation $\left(\frac{1}{\beta}\right)^\alpha \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$ with parameters $\alpha = \frac{1}{2}$ and $\beta = \frac{r_i^{i-1}}{\lambda}$. Relations 8 and 18 allow us therefore to write:

$$\phi(z_1, z_2, \dots, z_n) = A^n e^{-\lambda(z_1^2 + z_2^2 + \dots + z_n^2)} = A^n e^{-\lambda Z^2} \quad (21)$$

According to the relation 11, one obtains:

$$\frac{d^n N}{N} = A^n e^{-\lambda Z^2} dz_1 dz_2 \dots dz_n \quad (22)$$

If one focuses on the number dN_Z of vectors which have their arrowhead between the two hyperspheres of Z and $Z + dZ$ radius, one obtains thus:

$$\frac{dN_Z}{N} = A^n e^{-\lambda Z^2} \int \dots \int dz_1 dz_2 \dots dz_n \quad (23)$$

The equation above expresses the relative number of hydraulic length included between the hyperspheres of Z and $Z + dZ$ radius. Multiple integral expresses the volume included between these two infinitely close hyperspheres in a n -dimensions space. The element of volume dV is proportional to $Z^{n-1} dZ$:

$$\frac{dN_Z}{N} = A^n e^{-\lambda Z^2} B Z^{n-1} dZ \quad (24)$$

B being a constant. To clarify the variable L , which is the most interesting, one can make a change of variables. Since the relation 2 means that component l_i is on average r_l times larger than l_{i-1} , if we decide to argue about the mean values of z_i^2 , relations 5 and 6 give:

$$Z^2 = n\overline{z_i^2} \quad (25)$$

The hypothesis of isotropy permits in particular cases to write this relation for $i = 1$:

$$Z^2 = n\overline{z_1^2} = n\overline{x_1^2} = nL \left(\frac{r_1 - 1}{r_1^n - 1} \right) \quad (26)$$

Where L is the average hydraulic length corresponding to the vectors which end on the hypersphere of radius Z . Thus, the relation 21 enables to express the relative number of hydraulic lengths $\frac{dN_L}{N}$ whose value is between L and $L + dL$:

$$\frac{dN_L}{N} = C e^{-\frac{\mu}{2} L^{\frac{n}{2}-1}} dL \quad (27)$$

Where

$$\mu = \lambda n \left(\frac{r_1 - 1}{r_1^n - 1} \right) \quad (28)$$

μ and C being constants. The pdf of hydraulic lengths $\rho(L)$ can now be written:

$$\rho(L) = \frac{dN_L}{NdL} = C e^{-\mu L^{\frac{n}{2}-1}} \quad (29)$$

By integrating relation 27, L varying from 0 to ∞ , one finds C :

$$C = \frac{\mu^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \quad (30)$$

Where Γ is gamm function. The constant μ can be deducted from the average hydraulic length by complete following:

$$\overline{L} = \int_0^\infty L \rho(L) dL = \frac{n}{2\mu} \quad (31)$$

Which gives:

$$\mu = \frac{n}{2\overline{L}} \quad (32)$$

The pdf of hydraulic lengths $\rho(L)$ can be so written:

$$\rho(L) = \frac{dN_L}{NdL} = \left(\frac{n}{2\overline{L}} \right)^{\frac{n}{2}} \frac{1}{\Gamma\left(\frac{n}{2}\right)} L^{\frac{n}{2}-1} e^{-\frac{nL}{2\overline{L}}} \quad (33)$$

Let us remember that L is the hydraulic length, \overline{L} is the average of all the possible hydraulic lengths on the studied plant, n is the order of the tree, and that Γ is the gamma function. One can easily recognize in this last relation $f(L, \alpha, \beta)$ the gamma law with parameters $\alpha = \frac{n}{2}$ and $\beta = \frac{2\overline{L}}{n}$.

Order i	$R_C = \frac{N_{i-1}}{N_i}$	$R_L = \frac{L_i}{L_{i-1}}$
2	12.06	2.16
3	11.5	4.25
Theoretical	11.78	3.03

Table 1. average values of R_C and R_L calculated from the experimental data. The last values of the Table are calculated from a logarithmic regression between the order $i - 1$ and the large number of experimental values of R_C and R_L . The fractal dimension is thus $D = \frac{\ln(R_C)}{\ln(R_L)} = 2.22$.

3 Results and discussion

There is a big difference between systems involved in statistical physics and the system we use in our reasoning : Maxwell, for example, does not indicate the shape of the container that contains the N molecules because it does not matter. In our case, the studied plant can have very different shapes, its size is very variable and its order can vary from 1 for a young maiden tree to 8 or 9 for very big trees. Moreover, even though law 33 is very general, it will be all the more respected since the two conditions to apply a reasoning of statistical physics will be respected: a large number of elementary constituents (that is of hydraulic distances) and homogeneity of the population.

So one will either choose a big tree or a population of several trees with the same species having grown in the same environment. The pdf of hydraulic lengths is calculated through the law 33 by *i*) taking as the order n of the specified population value ν of the maximum order observed in the population, *ii*) taking as the average of hydraulic lengths \bar{L} calculated from N measures. We chose 12 apple trees (*Malus pumila* (L) Mill.) four years old and from the same "parents" and with order 3. The hydraulic lengths were measured manually from all the growing shoots.

The average values of the ratios R_C and R_L which are presented in the table 1 are calculated for all the sections. One can notice that the stability of R_C is excellent, but that R_L is a little more variable, as noted by many authors (Horton, 1945¹², Schumm, 1956⁵ and Shreve, 1967⁸). The stability of this last parameter, *a priori* considered to define the symbolic space is therefore verified. For each parameter, the accepted value is calculated by logarithmic regression because of the shape of R_C and R_L laws (values on the last Table line). By the way, one can note that the values of R_C and R_L lead to a fractal dimension $D = 2.22$. Figure 2 shows the components of hydraulic lengths as function of $\frac{l_i}{r_i^{D-1}}$. As one can see, considering the limited number of objects in some classes, the distribution law of components 20, as well as the isotropy hypothesis, from which it is deduced, can be considered as well verified. Figure 3 shows the theoretical graph supplied by the equation 33 and the experimental one obtained by measurement. Considering the very general hypotheses from which the theoretical graph is deduced, one can be struck by the fact that it coincides correctly with the experimental one.

Of course the theory and the experimental data do not fit as well as in statistical thermodynamics. For example, Maxwell's distribution in a molecular stream can be directly verified by counting the number of molecules that have a given velocity.

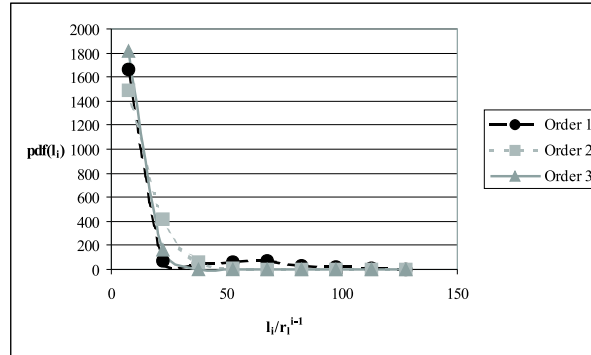


Figure 2. The pdf of the reduced hydraulic components l_i according to equation 20.

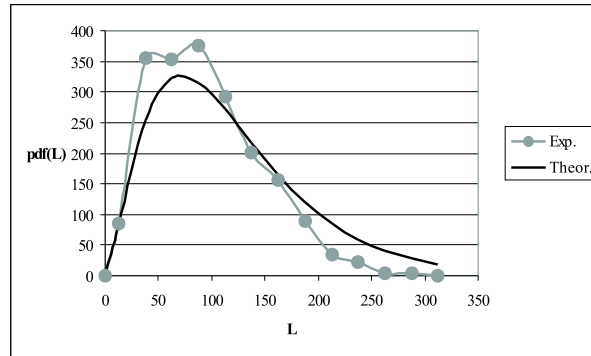


Figure 3. The pdf of the hydraulic lengths (Theor.) of the studied plant compared with the experimental results (Exp.). The pdf parameters are $n = 3$ and $\bar{L} = 47.6$ cm.

Miller and Kusch (1956)²⁵ showed that theoretical prediction was strikingly verified by the experiment. In thermodynamics, considering the huge number of microscopic objects, the relative fluctuations of the molecule numbers are proportional to $\frac{1}{\sqrt{N}}$, where N is the number of molecules in the considered class, so they become imperceptible at the macroscopic scale. In the case of botany, conformity cannot be rigorous because of two reasons which proceed directly from two conditions we had put *a priori* in order to apply a reasoning of statistical physics: *i*) the number of hydraulic lengths, corresponding to apexes, can not exceed a few thousand, or ten thousand, for a given class; thus the statistical fluctuations will always be much more important, compared to the fluctuations one can observe in thermodynamics, even if one can reduce them by widening the classes, *ii*) moreover, the distribution of hydraulic lengths, as well as the distribution of their n components can be more or less influenced by the environment constraints. We based our demonstration on the frame that Maxwell used to study the law of distribution of molecular velocities

in a gas. One knows that his theory was reused on other bases by Boltzmann and Gibbs. It would be interesting to apply general formalism of statistical mechanics to the botanical morphology.

4 Conclusion

We have just presented an original reasoning of statistical physics as far as it applies to a macroscopic object made up of elements themselves macroscopic: the plant. The study of the spatial organization of a plant leads to a mathematical description that completes, through the pdf of the hydraulic lengths, a classical morphogenetic description of its architecture. Such a description is essential in botany, whether it is for the understanding of the functioning of the plant or for the landscape analysis. Moreover, we think that the innovative approach adopted here the introduction of a fractal description into a reasoning of statistical physics could be applied successfully to other physical domains. The application of our results to other branched objects such as river systems or vascular systems can be easily attempted. One can also try to apply them to other objects provided they are fractal as for example a fractured surface in a solid material test.

Besides the interest of a morphological description of the tree, the above theory opens many perspectives of application in the dynamic domain. The mathematical description of the branches' organization indeed allows us to describe the dynamics of the transfers which take place in the plant on one hand and in the modeling of the dynamics of the plant genesis on the other hand. Thus the applications could be immediate in the fields of the complete plant physiology and in the ecophysiology, for example in the mechanism of axes selection. In particular, we shall show in another article Raimbault *et al.* (2001)³ how function 33 established above can be connected with morphogenesis concepts as the apical control.

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